ALFVENIC SOLITONS IN ULTRARELATIVISTIC ELECTRON-POSITRON PLASMAS

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Abstract. In electron-positron plasmas some of the plasma modes are decoupled due to the equal charge-to-mass ratio of both species. We derive the dispersion law for a low-frequency, generalized X-mode, which exists at all angles of propagation with respect to the static magnetic field. Its nonlinear evolution is governed by a Korteweg-de Vries equation, valid at all angles of propagation except strictly parallel propagation, for which a different approach leads to a vector form of the modified Korteweg-de Vries equation. The nonlinearity is strongest at perpendicular propagation. Ultrarelativistic effects are discussed,

Key words: Electron-positron plasmas - Pulsars - Nonlinear modes

1. Introduction

Electromagnetic waves in relativistic plasmas pertaining to magnetospheres of pulsars and active galactic nuclei have been investigated rather intensively in the literature (Sturrock, 1971; Max and Perkins, 1972; Max, 1973; Kennel and Pellat, 1976; Buti, 1978; Sweeney and Stewart, 1978; Lakhina and Buti, 1981; Asseo, 1984; Lominadze *et al.*, 1983; Shukla, 1985; Stenflo *et al.*, 1985; Shukla *et al.*, 1986; Lakhina and Tsintsadze, 1990; Shukla and Stenflo, 1993). Large-amplitude Alfvén waves propagating parallel to the external magnetic field in relativistic electron-positron have been studied by Sakai and Kawata (1980 a, b), Mikhailovskii *et al.* (1985 a,b,c) and Verheest (1996). The selfconsistent analysis of Verheest (1996) showed that the evolution of nonlinear parallel Alfvén modes is described by a vector form of a modified Korteweg-de Vries (mKdV) equation.

Since large-amplitude, low-frequency modes are most suited for the acceleration of charged particles to very high energies (Kennel and Pellat, 1976; Asseo, 198-1; Lakhina and Tsintsadze, 1990), it is of practical interest to

" *On* leave of absence from Indian Institute of Geomagnetism, Dr Nanabhai Moos Marg, Colaba, Mumbai 400005, India investigate also the nonlinear evolution of large amplitude obliquely propagating modes. In a previous paper (Verheest and Lakhina, 1996), we studied large-amplitude Alfvén modes propagating at an arbitrary angle to the external magnetic field in a cold, relativistic electron-positron plasma. We now include also pressure and ultrarelativistic effects, for both cases of parallel and oblique (including perpendicular) propagation.

Because in pulsars the magnetic field can be large, both the gyrofrequencies and the Alfvén velocity can be comparable to the plasma frequencies or the speed of light, respectively. Although in the initial phase of our investigations we have not imposed a particular polarization for the nonlinear waves under study, the low-frequency reductive perturbation analysis automatically picks out the generalized extraordinary (X) mode which we discuss.

The paper is organized as follows. in Section 2 we delineate the theoretical model and briefly discuss the linear modes in a magnetized electron-positron plasma, propagating at arbitrary angles to the static field. A low-frequency generalized X-mode is decoupled from the other modes, at all angles, the nonlinear evolution of which is given in Section 3 for parallel propagation and in Section 4 for oblique propagation. In the parallel case one obtains a nonintegrable vector form of the mKdV equation, whereas truly oblique propagation is governed by a KdV equation. Finally, our conclusions are formulated in Section 5.

2. Theoretical model and linear modes

Our model is that of a plasma immersed in a uniform magnetic field, and composed of equal numbers of electrons and positrons. We will look at phenomena propagating along the z–axis, so that all quantities depend only on z and t. To include also the possibility of oblique modes, we take $\mathbf{B}_0 = B_0\mathbf{e}_B$, with $\mathbf{e}_B = \sin\vartheta\,\mathbf{e}_x + \cos\vartheta\,\mathbf{e}_z$ the unit vector along the static magnetic field. The basic fluid equations (Mikhailovskii *et al.*, 1985b) include the continuity equations,

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial z}(n_\alpha u_{\alpha z}) = 0, \tag{1}$$

and the equations of motion,

$$\frac{\partial}{\partial t}(\rho_\alpha \mathbf{u}_\alpha) + \frac{\partial}{\partial z}(\rho_\alpha u_{\alpha z}\mathbf{u}_\alpha) = -\frac{\partial P_\alpha}{\partial z}\mathbf{e}_z \mp e n_\alpha\left(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\right). \tag{2}$$

Here E and B are the electric and magnetic fields, and the label α characterises the electrons ($\alpha = e$, with charge $q_e = -e$ and upper signs in the equations) and the positrons ($\alpha = p$, with $q_p = e$ and lower signs), with densities n_α, mass densities ρ_α, pressures P_α and velocities $\mathbf{u}_\alpha$. For the mass densities, defined as

$$\rho_\alpha = \frac{P_\alpha + \mathcal{E}_\alpha}{c^2 - u_\alpha^2}, \tag{3}$$

where $\mathcal{E}_\alpha$ is the mass energy density, and the pressures in an ultra relativistic description we need in addition

$$\frac{\partial \rho_\alpha}{\partial t} + \frac{\partial}{\partial z}(\rho_\alpha u_{\alpha z}) = \frac{1}{c^2}\left(\frac{\partial P_\alpha}{\partial t} \mp e n_\alpha \mathbf{E}\cdot\mathbf{u}_\alpha\right), \tag{4}$$

$$P_\alpha = C\left(n_\alpha\sqrt{1 - \frac{u_\alpha^2}{c^2}}\right)^\gamma . \tag{5}$$

The description is closed by Maxwell's equations

$$\mathbf{e}_z \times \frac{\partial \mathbf{E}}{\partial z} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}, \tag{6}$$

$$c^2\mathbf{e}_z \times \frac{\partial \mathbf{B}}{\partial z} = \frac{\partial \mathbf{E}}{\partial t} + \frac{e}{\varepsilon_0}(n_p\mathbf{u}_p - n_e\mathbf{u}_e), \tag{7}$$

$$\frac{\partial E_z}{\partial z} = \frac{e}{\varepsilon_0}(n_p - n_e). \tag{8}$$

Before going to the nonlinear development, we look at linear wave propagation, by linearizing and Fourier transforming the relevant equations (l)-(8). This yields an intricate wave equation, which can be split into two different parts, however. One of these corresponds to an electromagnetic wave with electric field perpendicular to the plane containing both the wave vector **k** and the static field B_0, decoupled from the other wave phenomena. This mode has dispersion law

$$\begin{aligned}&\omega^6 - \omega^4(\omega_p^2 + \Omega^2 + k^2c^2 + k^2c_s^2)\\ &+\omega^2(k^2c^2\Omega^2 + k^2c_s^2\omega_p^2 + k^2c_s^2\Omega^2 \cos^2\vartheta + k^4c^2c_s^2)\\ &- k^4c^2c_s^2\Omega^2\cos^2\vartheta = 0.\end{aligned} \tag{9}$$

The relativistic plasma frequency ω_p is defined through $\omega_p^2 = 2N^2e^2/\varepsilon_0\rho_0$ *(see* Sakai and Kawata, 1980 b), taking electrons and positrons together, with $N = N_e = N_p$ the equilibrium density and ρ_0 the equilibrium mass density. Similarly, the relativistic gyrofrequency is for both species $\Omega = eNB_0/\rho_0$, in absolute value. Finally, the thermal velocity c_s is given through $c_s^2 = \gamma P_0/\rho_0$.

It turns out that (9) has a high-frequency as well as a low-frequency branch. For the latter, the long-wavelength limit shows that the linear phase velocity V obeys

$$\begin{aligned}&V^4(\omega_p^2 + \Omega^2) - V^2(c^2\Omega^2 + c_s^2\omega_p^2 + c_s^2\Omega^2\cos^2\vartheta)\\ &+c^2c_s^2\Omega^2\cos^2\vartheta = 0.\end{aligned} \tag{10}$$

In the limit of parallel propagation ($\vartheta = 0$) one obtains (besides a spurious root) the Alfvén mode, with

$$V^2 = V_A^2 = \frac{c^2\Omega^2}{\omega_p^2 + \Omega^2}, \tag{11}$$

showing the proper definition for the Alfvén velocity in an electron-positron plasma. For perpendicular propagation (ϑ = 900), (10) reduces to

$$V^2 = V_A^2 + \frac{c_s^2\omega_p^2}{\omega_p^2 + \Omega^2}, \tag{12}$$

giving the fast magnetosonic velocity (together with a spurious root zero). This tallies with recent work on magnetosonic modes in multispecies dusty plasmas (Meuris and Verheest, 1996), if results are specialized to equal-mass plasmas.

Since the mode described by (9) corresponds at perpendicular propagation to part of the X–mode, but exists as a separate entity at all angles of propagation, we call it a generalized X–mode. In ordinary plasmas the X--mode cannot be factorized, and for oblique propagation all modes are mixed together.

The other components of E, those in the plane spanned by k and B_o, obey a complicated dispersion law, which we shall not discuss further.

3. Nonlinear evolution: parallel propagation

As the dispersion law (9) only contains even powers of k and ω, there is a quadratic correction in k to the linear phase velocity, resulting in the standard Korteweg-de Vries (Kd V) stretching

$$\xi = \varepsilon(z - Vt), \qquad \tau = \varepsilon^3 t. \tag{13}$$

At $\vartheta = 0$ we will use for all variables an expansion of the type

$$f = F + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \tag{14}$$

Substitution of the stretching (13) and the perturbation expansions (14) into the equations (l)–(8) gives a sequence of equations, upon equating the coefficients of the various powers of ε. To lowest nonzero order we find for the parallel-type quantities that

$$n_{e2} = n_{p2} = N\frac{V^2(c^2 - c_s^2)}{2c^2(V^2 - c_s^2)}\frac{B_{\perp 1}^2}{B_0^2},$$

$$p_{e2} = p_{p2} = \rho_0\frac{V^2(V^2 + c^2)(c^2 - c_s^2)}{2c^4(V^2 - c_s^2)}\frac{B_{\perp 1}^2}{B_0^2},$$

$$
\begin{aligned}
P_{e2} = P_{p2} &= \rho_0 c_s^2 \frac{V^2(c^2 - V^2)}{2c^2(V^2 - c_s^2)} \frac{B_{\perp 1}^2}{B_0^2}, \\
u_{e\|2} = u_{p\|2} &= \frac{V^3(c^2 - c_s^2)}{2c^2(V^2 - c_s^2)} \frac{B_{\perp 1}^2}{B_0^2},
\end{aligned}
\tag{1.5}
$$

showing that to this order there is charge neutrality, without it having been assumed from the outset. Turning now our attention to the perpendicular variables leads to

$$
\begin{aligned}
\mathbf{u}_{\alpha\perp 1} &= -\frac{V}{B_0}\mathbf{B}_{\perp 1}, \\
\mathbf{u}_{\alpha\perp 2} &= -\frac{V}{B_0}\mathbf{B}_{\perp 2} \mp \frac{V^2}{\Omega B_0}\mathbf{e}_z \times \frac{\partial \mathbf{B}_{\perp 1}}{\partial \xi}, \\
\mathbf{u}_{\alpha\perp 3} &= \frac{1}{B_0}\mathbf{E}_{\perp 3} \times \mathbf{e}_z + \frac{V^3(c^2 - c_s^2)}{2c^2(V^2 - c_s^2)B_0^3} B_{\perp 1}^2 \mathbf{B}_{\perp 1} \\
&\quad \mp \frac{V^2}{\Omega B_0}\mathbf{e}_z \times \frac{\partial \mathbf{B}_{\perp 2}}{\partial \xi} + \frac{V^3}{\Omega^2 B_0}\frac{\partial^2 \mathbf{B}_{\perp 1}}{\partial \xi^2},
\end{aligned}
\tag{16}
$$

having used Faraday's equation (6). Inserting all obtained expressions into the perpendicular components of Ampère's equation (7), we find to orders 2 and 3 that it indeed vanishes due to the definition (11) of the phase velocity *v.* To order 4 we obtain the nonlinear evolution equation

$$
\begin{aligned}
\frac{4}{V}\left(1 + \frac{\Omega^2}{\omega_p^2}\right)^2 \frac{\partial}{\partial \tau}\mathbf{B}_{\perp 1} &+ \frac{V^2(c^2 - c_s^2)}{c^2(V^2 - c_s^2)B_0^2}\frac{\partial}{\partial \xi}\left(B_{\perp 1}^2 \mathbf{B}_{\perp 1}\right) \\
&+ \frac{2c^2}{\omega_p^2}\frac{\partial^3}{\partial \xi^3}\mathbf{B}_{\perp 1} = \mathbf{0}.
\end{aligned}
\tag{17}
$$

This looks like a cross between a vector version of an mKdV equation and a DNLS equation, and has been discussed in the cold-plasma limit before (Verheest, 1996). For linear polarization, where $\mathbf{B}_{\perp 1}$ retains a fixed direction, we recover the mKdV equation given by Mikhailovskii *et al.* (1985 b), although other polarizations are possible (Verheest, 1996). When projecting (17) out in normalized form, with all coefficients unity, and going over to a complex representation of the wave magnetic field $\phi = B_{x1} + iB_{y1}$, we find the complex mKdV equation (Karney *et al.*, 1978), which is in general not integrable.

4. Nonlinear evolution: oblique and perpendicular propagation

At truly oblique propagation ($\vartheta \neq 0$ and $\sin\vartheta$ finite) the generalized X-mode is such that to all orders the electron and positron quantities obey

$$
\begin{aligned}
n_e &= n_p, & \rho_e &= \rho_p, & P_e &= P_p, \\
u_{ex} &= u_{px}, & u_{ey} &= -u_{py}, & u_{ez} &= u_{pz}.
\end{aligned}
\tag{18}
$$

This means that we can restrict ourselves to the positron quantities from the outset and drop the species index for the sake of brevity. Analogous results hold for the wave fields, with

$$
\begin{aligned}
&E_x = E_z = 0, \\
&B_y = 0, \qquad B_z = B_{\|0}.
\end{aligned}
\tag{19}
$$

Thus we need only consider the following set of fluid equations,

$$
\begin{aligned}
&\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(n u_z) = 0, \\
&\frac{\partial}{\partial t}(\rho u_x) + \frac{\partial}{\partial z}(\rho u_x u_z) = enB_{\|0} u_y, \\
&\frac{\partial}{\partial t}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_y u_z) = en(E_y + u_z B_x - u_x B_{\|0}), \\
&\frac{\partial}{\partial t}(\rho u_z) + \frac{\partial}{\partial z}(\rho u_z^2) = -\frac{\partial P}{\partial z} - enB_x u_y, \\
&\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u_z) = \frac{1}{c^2}\left(\frac{\partial P}{\partial t} + enE_y u_y\right), \\
&P = C\left(n\sqrt{1 - \frac{u^2}{c^2}}\right)^{\gamma}.
\end{aligned}
\tag{20}
$$

together with the nonzero Maxwell's equations

$$
\begin{aligned}
&\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}, \\
&c^2 \frac{\partial B_x}{\partial z} = \frac{\partial E_y}{\partial t} + \frac{2en}{\varepsilon_0} u_y.
\end{aligned}
\tag{21}
$$

Our generalized X-mode is characterized by zero currents and electric fields in the plane containing the static magnetic field and the direction of propagation, and also by strict quasi-neutrality, due to the low-frequency approach, but not really assumed to begin with.

As now the usual KdV expansions are adhered to (see the discussion in the cold-plasma case, Verheest and Lakhina, 1996), we take

$$
\begin{aligned}
n &= N + \varepsilon^2 n_2 + \varepsilon^4 n_4 + \cdots, \\
\rho &= \rho_0 + \varepsilon^2 \rho_2 + \varepsilon^4 \rho_4 + \cdots, \\
P &= P_0 + \varepsilon^2 p_2 + \varepsilon^4 p_4 + \cdots, \\
u_x &= \varepsilon^2 u_{x2} + \varepsilon^4 u_{x4} + \cdots, \\
u_y &= \varepsilon^3 u_{y3} + \varepsilon^5 u_{y5} + \cdots, \\
u_z &= \varepsilon^2 u_{z2} + \varepsilon^4 u_{z4} + \cdots, \\
B_x &= B_{\perp 0} + \varepsilon^2 B_{x2} + \varepsilon^4 B_{x4} + \cdots, \\
E_y &= \varepsilon^2 E_{y2} + \varepsilon^4 E_{y4} + \cdots
\end{aligned}
\tag{22}
$$

Substitution of the stretching (13) and the perturbation expansions (22) into the equations (20)-(21) now gives to lowest nonzero order that

$$
\begin{aligned}
n_2 &= \frac{N V^2 B_{x2} \sin\vartheta}{B_0(V^2 - c_s^2 \cos^2\vartheta)}, \\
\rho_2 &= \frac{\rho_0 (c^2 + c_s^2) V^2 B_{x2} \sin\vartheta}{B_0 c^2 (V^2 - c_s^2 \cos^2\vartheta)}, \\
P_2 &= \frac{pot,^2 V^2 B_{x2} \sin\vartheta}{B_0(V^2 - c_s^2 \cos^2 2'?)}, \\
u_{x2} &= -\frac{V(V^2 - c_s^2) B_{x2} \operatorname{Cos}\vartheta}{B_0(V^2 - c{:}COS20)}, \\
u_{y3} &= \frac{V^2(V^2 - c_s^2)}{\Omega B_0(V^2 - c_s^2 \cos^2 r?)} \frac{\partial B_{x2}}{\partial z}, \\
u_{z2} &= \frac{V^3 B_{x2} \sin\vartheta}{B_0(V^2 - c_s^2 \cos^2\vartheta)}, \\
E_{y2} &= -V B_{x2},
\end{aligned}
\tag{23}
$$

and Ampère's law vanishes to order 3 in E, as now V obeys (12). Note in particular the relativistic correction in the mass density p_2.

Going further through the algebra to order 5, essentially along the lines of our previous paper (Verheest and Lakhina, 1996), we deduce a KdV equation

$$
A\frac{\partial B_{x2}}{\partial \tau} + C B_{x2}\frac{\partial B_{x2}}{\partial \xi} + D\frac{\partial^3 B_{x2}}{\partial \xi^3} = 0,
\tag{24}
$$

with coefficients

$$
\begin{aligned}
A &= \frac{1}{V^3}\left\{(V^2 - c_s^2\cos^2\vartheta)^2\frac{\Omega^2}{\omega_p^2} + (V^2 - c_s^2)^2\cos^2\vartheta + V4\sin2\vartheta\right\}, \\
C &= \frac{\sin\vartheta}{2B_0c^2(V^2 - c_s^2\cos^2\vartheta)}\{3(V^4 - 2V^2c_s^2 + c_s^4\cos^2\vartheta)(c^2 - c_s^2) \\
&\quad + V^2c_s^2\sin^2\vartheta[c^2(\gamma + 4) - 6c_s^2]\}, \\
D &= \frac{(V^2 - c_s^2)^2}{2\Omega^2}.
\end{aligned}
\tag{25}
$$

On taking $c_s^2 = 0$, i.e., under the cold plasma limit, (24) becomes identical to the KdV equation derived by Verheest and Lakhina (1996). At perpendicular propagation, the coefficients A and D can be derived from the corresponding expressions in nonrelativistic multispecies plasmas (Meuris and Verheest, 1996), when specializing those to equal-mass plasmas. However, C reads in the perpendicular case as

$$
C = \frac{1}{2B_0}\left\{3V^2\left(1 - \frac{c_s^2}{c^2}\right) + (\gamma - 2)c_s^2\right\},
\tag{26}
$$

and there is a relativistic correction (in c_s^2/c^2), which can become quite important in the ultra relativistic limit.

5. Conclusions

In Sections 3 and 4, we have derived the evolution equations for the parallel propagating Alfvén modes and the obliquely propagating generalized X-modes. We shall now discuss the implications of the ultrarelativistic limit on the solution of these evolution equations in some simple cases.

Let us first discuss the special case of linear polarization for which (17) reduces to the mKd V equation of Mikhailovskii *et al.* (1985b). Such mKdV actuations only have super-Alfvénic soliton solutions of sech-type, provided the coefficient of the nonlinear term is positive, since the other terms already have positive coefficients. Looking at (17), one sees that $c^* - c_s^2$ and $V^* - c_s^2$ should have the same sign for solitons to exist. Since $c^2 > V^*$ from (11), this implies either the ordering $c^2 > V^2 > c_s^2$ for supersonic relativistic solitons, or $c_s^2 > c^* > V^*$ for subsonic ultrarelativistic solitons. Indeed, the ultrarelativistic limit corresponds to taking $P_0/\mathcal{E}_0 >> 1$ such that the pressure due to ultra-high temperatures greatly exceeds the rest mass energy density of electrons or positrons. Such a limit appears to be relevant for the conditions existing in pulsar magnetospheres (Sturrock, 1971; Kennel and Pellat, 1976). In the ultrarelativistic limit, $c_s^2 \simeq \gamma c^2$. Note that $\gamma = 1$ corresponds to the isothermal case, and 5/3 to an adiabatic case. This means that in the ultrarelativistic limit (and $\gamma > 1$), supersonic Alfvén solitons cannot exist, but linearly polarized subsonic Alfvén solitons are possible, although they are otherwise not allowed in relativistic plasmas, as had already been noted by Mikhailovskii *et al.* (1985b). In addition, there is a forbidden regime when no solitons at all can exist, namely $c^2 > c_s^2 > V^2$, shedding another light on the transition to ultrarelativistic electron-positron plasmas. For a further discussion of possible values for γ, see the recent paper by Gratton *et al.* (1997).

For the oblique modes, we note from (24) and (25) that the coefficients A and D are positive definite, irrespective of whether the plasma is ultrarelativistic or not. For perpendicular propagation, $C > 0$ for cold relativistic plasmas, but $C < 0$ in the ultrarelativistic limit for all reasonable values $1 \leq \gamma$ as seen from (26), provided that $\Omega^2 < 2\omega_p^2$. On the other hand, for $\Omega^2 >> \omega_p^2$ we find $C < 0$ for $1 \leq \gamma < 4.3$. Hence in ultrarelativistic electron-positron plasmas perpendicularly propagating Alfvén solitons of sech^2-type have a different nature (compressive/rarefactive, although these terms are not really appropriate for the changes in magnetic field here) compared to cold relativistic plasmas (Verheest and Lakhina, 1996). In addition, there is also the possibility in the ultra relativistic case that the coefficient C can be positive for some oblique angles, different from exact perpendicular propagation. Then (24) would permit the existence of oblique Alfvén solitons of the same nature as in cold relativistic plasmas.

The nonlinear Alfvén solitons studied here would be relevant for interpreting the microstructure in the pulsar radiation or subpulses. One of the mechanisms for pulsar radiation is the synchrotrons radiation produced by bunches of electron/positron streaming along the magnetic field lines. The subpulses can arise by the modulation of high-frequency synchrotrons radiation by low-frequency Alfvén solitons. Interaction of Alfvén solitons with electrons can lead to heating and acceleration of electrons to cosmic ray energies.

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References

Asseo, E.: 1984, 'Large amplitude electromagnetic waves' in McNamara, B., cd(s)., *Radiation in Plasmas,* World Scientific: Singapore, vol. I, 39-69

Buti, B.: 1978, 'Relativistic Langmuir solitons -- application to pulsars', *Astrophys. Space Sci.* 58, 89-97

Gratton, F. T., Gnavi, G., Galvão, R.M. O. and Gomberoff, L.: 1997, 'Self-modulation of a strong electromagnetic wave in a positron- electron plasma induced by relativistic temperatures and phonon damping', *Phys. Rev. E* 55, 3381-3392

Karney, C. F. F., Sen, A. and Chu, F. Y. F.: 1978, 'The complex modified Korteweg-de Vries equation, a non-integrable evolution equation' in Bishop, A.R. and Schneider, T., cd(s)., *Solitons and Condensed Matter Physics*, Springer: Berlin, 71-75

Kennel, C.F. and Pellat, R,: 1976, 'Relativistic nonlinear plasma waves in a magnetic field', *J. Plasma Phys.* 15, 335-355

Lakhina, G.S. and Buti, B.: 1981, 'Generation of a d.c. field by nonlinear electromagnetic waves in relativistic plasmas', *Astrophys. Space Sci.* 79, 25-36

Lakhina, G.S. and Tsintsadze, N.L.: 1990, 'Large-amplitude low-frequency electromagnetic waves in pulsar magnetospheres', *Astrophys. Space Sci.* **174**, 143-1s0

Lominadze, J. G., Machabeli, G.Z. and Usov, V. V.: 1983, 'Theory of NP 0532 pulsar radiation and the nature of the activity of the Crab nebula', *Astrophys. Space* Sci. 90, 19-43

Max, C. E.: 1973, 'Steady-state solutions for relativistically strong electromagnetic waves in plasmas', *Phys. Fluids* 16, 1277-1288

Max, C.E. and Perkins, F. W.: 1972, 'Instability of a relativistically strong electromagnetic wave of circular polarization', *Phys. Rev. Lett. 29,* 1731-1734

Meuris, P. and Verheest, F.: 1996, 'Korteweg- de Vries equation for magnetosonic modes in dusty plasmas', *Phys. Lett. A 219, 299-302*

Mikhailovskii, A, B., Onishchenko, O.G. and Smolyakov, A. I.: 1985a, 'Theory of low-frequency electromagnetic solitons in a relativistic electron-positron plasma', *Sov. J. Plasma Phys.* 11, 215-219

Mikhailovskii, A. B., Onishchenko, O.G. and Tatarinov, E.G.: 1985b, 'Alfvén solitons in a relativistic electron-positron plasma, I: Hydromagnetic theory', *Plasma Phys. Contr. Fusion* 27, 527-537

Mikhailovskii, A. B., Onishchenko, O.G. and Tatarinov, E. G.: 1985c, 'Alfvén solitons in a relativistic electron-positron plasma, 11: Kinetic theory', *Plasma Phys. Contr.* Fusion 27, 539-556

Sakai, J. and Kawata, T.: 1980a, 'Wavesin an ultra-relativistic electron-positron plasma', *J. Phys. Sot. Japan* 49, 747-752

Sakai, J. and Kawata, T.: 1980b, 'Nonlinear Alfvén waves in an ultra-relativistic electron-positron plasma', *J. Phys. Sot. Japan* 49, 753-758

Shukla, P. K.: 1985, 'Self-modulation of pulsar radiation in strongly magnetized electron-positron plasmas', *Astrophys. Space Sci.* 114, 381-385

Shukla, P. K., Rae, N. N., Yu, M.Y. and Tsintsadze, N. L.: 1986, 'Relativistic nonlinear effects in plasmas', *Phys. Rep.* 138, 1-149

Shukla, P.K. and Stenflo, L.: 1993, 'Nonlinear coupling between electromagnetic fields in a strongly magnetized electron-positron plasma', *Astrophys. Space* Sci. 209, 323-326

Stenflo, L., Shukla, P.K. and Yu, M. Y.: 1985, 'Nonlinear propagation of electromagnetic waves in magnetized electron-positron plasmas', *Astrophys. Space Sci.* 117, 303-308

Sturrock, P. A.: 1971, 'A model of pulsars', *Astrophys. J.* 164, 529-556

Sweeney, G.S.S. and Stewart, P.: 1978, 'Magneto-parametric instabilities in the Crab Nebula: 11', *Astron. Astrophys.* 66, 139–153

Verheest, F.: 1996, 'Solitary Alfvén modes in relativistic electron-positron plasmas', *Phys. Lett. A* 213, 177-182

Verheest, F. and Lakhina, G. S.: 1996, 'Oblique solitary Alfvén modes in relativistic electron-positron plasmas', *Astrophys. Space Sci.* 240, 215-224